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Superdiffusivity of Two Dimensional Lattice Gas Models

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It was proved [Navier–Stokes Equations for Stochastic Particle System on the Lattice. *Comm. Math. Phys*. (1996) 182, 395; Lattice gases, large deviations, and the incompressible Navier–Stokes equations. *Ann. Math.* (1998) 148, 51] that stochastic lattice gas dynamics converge to the Navier–Stokes equations in dimension $d=3$ in the incompressible limits. In particular, the viscosity is finite. We proved that, on the other hand, the viscosity for a two dimensional lattice gas model diverges faster than $\log\log t$. Our argument indicates that the correct divergence rate is $(\log t)^{1/2}$. This problem is closely related to the logarithmic correction of the time decay rate for the velocity auto-correlation function of a tagged particle.

KEY WORDS: Hydrodynamic limit; second class particle; superdiffusivity.

1. INTRODUCTION

It is well-known that although the classical dynamics is time reversible, the macroscopic behavior of the fluid, governed by the Navier–Stokes equations, is time irreversible. The measure on the time irreversibility is characterized by the viscosity, also called the bulk diffusivity of the system. It can be represented as the diffusion coefficient of a second class particle in the fluid. Instead of a second class particle, one can study the typical behavior of a tagged particle. Once again, even though the underlying

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dynamics is time reversible, the tagged particle is diffusive. The diffusion coefficient in this case is called the self diffusion coefficient. The bulk and self diffusion coefficients are two different quantities, but they share similar qualitative behavior.

The Green–Kubo formulae represent the bulk or self diffusion coefficients as time integrals over the current correlation function or the velocity correlation function. In the fundamental work of Alder and Wainwright,[1] it predicts that the time decay of the velocity correlation function is of order $t^{-d/2}$, here d is the dimension of the system. Since the decay is $1/t$ in $d=2$ is not integrable, it predicted that the self-diffusion coefficient of the two dimensional fluid diverges. In the later work, Alder *et al.*[2] proposed that the decay in two dimension is actually $t^{-1}\log t^{-1/2}$. However, the logarithmic correction cannot be seen from the their simulation. This decay rate was also obtained by Forster *et al.*[5] by a renormalization group method. The later simulation by van der Hoef and Frenkel[6] confirmed that there is a discrepancy to the pure algebraic decay t^{-1}, but is far from being able to determine the precise logarithmic correction.

If we formally integrate the law $t^{-1}(\log t)^{-1/2}$, we obtain from the Green–Kubo formula that the diffusion coefficient diverges as $(\log t)^{1/2}$ in dimension $d=2$. For higher dimension, the diffusion coefficient is expected to be finite. This was proved rigorously in various settings. For the stochastic lattice gas models considered in Ref. 4, the bulk diffusion coefficient is proved to be finite for $d \geqslant 3$. One key ingredient of these lattice gas models is the asymmetric simple exclusion process. For this process, both the bulk and self diffusion coefficients were proved to be finite in Refs. 12 and 14 for dimension $d \geqslant 3$. Van Beijeren *et al.*[3] predicted via the mode-coupling theory that the diffusivity diverges as $(\log t)^{2/3}$ in $d=2$ and $t^{1/3}$ for $d=1$. The two dimensional case was proved in Ref. 15; the one dimensional case was only partly solved.[11] However, related problems in the one dimensional case was solved by integrable method.[7]

In this paper, we shall proved that for the stochastic lattice model, the bulk diffusion coefficient diverges in $d=2$. Following the method of Ref. 11, we derive a series of upper and lower bounds to the diffusivity in terms of continuous fractions for operators. If we take the first lower bound to the diffusivity, we obtain the divergence rate in Theorem 2.1. If one assumes that the dispersion laws of these two sequences (upper and lower) of the continuous fractions converge, the divergence law $(\log t)^{1/2}$ can be obtained heuristically. See the discussion at the end of this paper. Notice that the exponent $1/2$ is different from the $2/3$ in the case of the asymmetric simple exclusion process. Since the lattice gas models are very complicated, we do not know if the argument of Ref. 15 can be extended to this case.

2. THE MODEL

We recall the lattice gas models considered in Ref. 4 in dimension d. Denote by $\{e_j, j=1,\dots,d\}$ the canonical basis of $\mathbb{R}^d$ and let $\mathcal{E}=\{\pm e_1,\dots,\pm e_d\}$. Let $\mathcal{V}\subset\mathbb{R}^d$ be a finite set representing the possible velocities. On each site of the lattice at most one particle for each velocity is allowed. A configuration of particles on the lattice is denoted by $\eta=\{\eta_x,\ x\in\mathbb{Z}^d\}$ where $\eta_x=\{\eta(x,v),\ v\in\mathcal{V}\}$ and $\eta(x,v)\in\{0,1\},\ x\in\mathbb{Z}^d,\ v\in\mathcal{V}$, is the number of particles with velocity v at x. The set of particle configurations is $X=\left(\{0,1\}^{\mathcal{V}}\right)^{\mathbb{Z}^d}$.

The dynamics consists of two parts: Asymmetric random walk with exclusion among particles of the same velocity and binary collisions between particles of different velocities. We first describe the random walk part of the dynamics. Particles of velocity v perform a continuous time asymmetric random walk with simple exclusion. A particle at x waits a random, exponentially distributed time then chooses a nearby site $x+y$ according to a certain jump law and jumps there as long as the site is not occupied by another particle of the same velocity. If there is a particle of the same velocity, the jump is suppressed and the particle waits for a new exponential time. All particles are doing this simultaneously, and since time is continuous ties do not occur. The jump law and waiting time are chosen so that the rate of jumping from site x to site $x+y$ is $p(y,v)$ which should be finite range, irreducible and have mean velocity v:

$$\sum_y y\, p(y,v) = v\,.$$

For the sake of concreteness, we take in this paper $p(y,v)=0$ unless $|y|=\sum_{1\leqslant j\leqslant d}|y_j|=1$ in which case

$$p(\pm e_j, v) = \gamma \pm (1/2)e_j\cdot v$$

for each vector e_j and some $\gamma>0$ large enough for all rates $p(e,v)$ to be non-negative. The generator $\mathcal{L}^{ex}$ of the random walk part of the dynamics acts on local functions f on the configuration space X by

$$(\mathcal{L}^{ex} f)(\eta) = \sum_{\substack{v\in\mathcal{V}\\ e\in\mathcal{E}}}\sum_{x\in\mathbb{Z}^d} p(x,e,v;\eta)\,[f(\eta^{x,x+e,v}) - f(\eta)]\,,$$

where

$$p(x,e,v;\eta) = \eta(x,v)\,[1-\eta(x+e,v)]\,p(e,v)$$

is the jump rate from x to $x+e$ for particles with velocity v and

$$\eta^{x,y,v}(z,w) = \begin{cases} \eta(y,v) & \text{if } w=v \text{ and } z=x, \\ \eta(x,v) & \text{if } w=v \text{ and } z=y, \\ \eta(z,w) & \text{otherwise.} \end{cases}$$

The collision part of the dynamics is described as follows. Denote by $\mathcal{Q}$ a collision set which preserves momentum:

$$\mathcal{Q} \subset \{(v,w,v',w') \in \mathcal{V}^4 : v+w = v'+w'\}.$$

Assume that $\mathcal{Q}$ is symmetric in the sense that (v,w,w',v'), (v',w',v,w), and (v',w',w,v) belong to $\mathcal{Q}$ as soon as (v,w,v',w') belongs to $\mathcal{Q}$. Particles of velocities v and w at the same site collide at rate one and produce two particles of velocities v' and w' at that site. The generator $\mathcal{L}^c$ is therefore

$$\mathcal{L}^c f(\eta) = \sum_{y \in \mathbb{Z}^d} \sum_{q \in \mathcal{Q}} p(y,q,\eta)\,[f(\eta^{y,q}) - f(\eta)],$$

where the rate $p(y,q,\eta)$ is given by

$$p(y,q,\eta) = \eta(y,v)\,\eta(y,w)\,[1-\eta(y,v')][1-\eta(y,w')]$$

and, for $q=(v_0,v_1,v_2,v_3)$, the configuration $\eta^{y,q}$ after the collision is defined as

$$\eta^{y,q}(z,u) = \begin{cases} \eta(y,v_{j+2}) & \text{if } z=y \text{ and } u=v_j \text{ for some } 0 \leqslant j \leqslant 3, \\ \eta(z,u) & \text{otherwise,} \end{cases}$$

where the sum in v_{j+2} should be understood modulo 4.

The generator $\mathcal{L}$ of the lattice gas we examine in this article is the superposition of the exclusion dynamics with the collisions just introduced:

$$\mathcal{L} = \mathcal{L}^{ex} + \mathcal{L}^c.$$

Let $\{\eta(t) : t \geqslant 0\}$ be the Markov process with generator $\mathcal{L}$.

2.1. The Invariant States

We assume that the sets $\mathcal{V}$ and $\mathcal{Q}$ are chosen in such a way that the unique conserved quantities are the local mass I_0 and local momentum I_a, $a=1,\dots,d$:

$$I_0(\eta_x) = \sum_{v\in\mathcal{V}} \eta(x,v)\,, \quad I_a(\eta_x) = \sum_{v\in\mathcal{V}} (v\cdot e_a)\,\eta(x,v)\,.$$

Examples of sets $\mathcal{V}$ and collision dynamics with this property are easy to produce. Consider, for example, $\mathcal{V}=\mathcal{E}$ and take $\mathcal{Q}$ as the set of all vectors (v,w,v',w') such that $v+w=v'+w'=0$. An elementary computation shows that the unique conserved quantities are total mass and momentum. Ref. 4 presents another example in $d=3$.

For each chemical potential $\boldsymbol{\lambda}=(\lambda_0,\lambda_1,\dots,\lambda_d)$, denote by $m_{\boldsymbol{\lambda}}$ the measure on $\{0,1\}^{\mathcal{V}}$ given by

$$m_{\boldsymbol{\lambda}}(\xi) = \frac{1}{Z(\boldsymbol{\lambda})}\exp\left\{\sum_{a=0}^{d}\lambda_a I_a(\xi)\right\}\,, \tag{2.1}$$

where $Z(\boldsymbol{\lambda})$ is a normalizing constant, $I_0(\xi)=\sum_{v\in\mathcal{V}}\xi(v)$, $I_a(\xi)=\sum_{v\in\mathcal{V}}(v\cdot e_a)\xi(v)$ for $a=1,\dots,d$. Notice that $m_{\boldsymbol{\lambda}}$ is a product measure on $\{0,1\}^{\mathcal{V}}$, i.e., that the variables $\{\xi(v): v\in\mathcal{V}\}$ are independent under $m_{\boldsymbol{\lambda}}$.

Denote by $\mu_{\boldsymbol{\lambda}}$ the product measure on $(\{0,1\}^{\mathcal{V}})^{\mathbb{Z}^d}$ with marginals given by

$$\mu_{\boldsymbol{\lambda}}\{\eta: \eta(x,\cdot)=\xi\} = m_{\boldsymbol{\lambda}}(\xi)$$

for each ξ in $\{0,1\}^{\mathcal{V}}$ and x in $\mathbb{Z}^d$. $\mu_{\boldsymbol{\lambda}}$ is a product measure in the sense that the variables $\{\eta(x,v): x\in\mathbb{Z}^d, v\in\mathcal{V}\}$ are independent under $\mu_{\boldsymbol{\lambda}}$. A simple computation shows that $\mu_{\boldsymbol{\lambda}}$ is an invariant state for the Markov process with generator $\mathcal{L}$ for each $\boldsymbol{\lambda}$ in $\mathbb{R}_+\times\mathbb{R}^d$, that the generator $\mathcal{L}^c$ is symmetric with respect to $\mu_{\boldsymbol{\lambda}}$ and that $\mathcal{L}^{ex}$ has an adjoint $\mathcal{L}^{ex,*}$ in which $p(e,v)$ is replaced by $p^*(e,v)=p(-e,v)$.

The expected value of the density under an invariant state can be computed explicitly. Fix a vector v in $\mathcal{V}$ and define $\theta_v: \mathbb{R}^{d+1}\to\mathbb{R}_+$ as the expected value of $\eta(x,v)$ under $\mu_{\boldsymbol{\lambda}}$:

$$\theta_v(\boldsymbol{\lambda}) := E_{\mu_{\boldsymbol{\lambda}}}[\eta(x,v)] = \frac{\exp\left\{\lambda_0+\sum_{a=1}^{d}\lambda_a(v\cdot e_a)\right\}}{1+\exp\left\{\lambda_0+\sum_{a=1}^{d}\lambda_a(v\cdot e_a)\right\}}\,.$$

In this formula and below, for a probability measure μ, E_μ stands for the expectation with respect to μ. The expectation under the invariant state $\mu_{\boldsymbol\lambda}$ of the mass and momentum are given by

$$\rho(\boldsymbol\lambda) := E_{\mu_{\boldsymbol\lambda}}[I_0(\eta_x)] = \sum_{v\in\mathcal{V}} \theta_v(\boldsymbol\lambda)$$

$$u_a(\boldsymbol\lambda) := E_{\mu_{\boldsymbol\lambda}}[I_a(\eta_x)] = \sum_{v\in\mathcal{V}} (v\cdot e_a)\theta_v(\boldsymbol\lambda)\,.$$

$(\rho(\boldsymbol\lambda), u_1(\boldsymbol\lambda), \dots, u_d(\boldsymbol\lambda))$ is the gradient of the strictly convex function $\log Z(\boldsymbol\lambda)$. In particular, the map which associates the chemical potential $\boldsymbol\lambda$ to the vector of density and momentum $(\rho, \boldsymbol{u}) = (\rho, u_1, \dots, u_d)$ is one to one. Therefore, the chemical potential $\boldsymbol\lambda = (\lambda_0, \dots, \lambda_d)$ can be expressed in terms of $(\rho, \boldsymbol{u})$: there exist a subset $\mathfrak{A}$ of $\mathbb{R}_+ \times \mathbb{R}^d$ and functions $\Lambda_a : \mathfrak{A} \to \mathbb{R}$, $a = 0, \dots, d$, such that

$$\lambda_a = \Lambda_a(\rho(\boldsymbol\lambda), \boldsymbol{u}(\boldsymbol\lambda)) \tag{2.2}$$

for each $\boldsymbol\lambda$ in $\mathbb{R}_+ \times \mathbb{R}^d$. Let $\boldsymbol\Lambda = (\Lambda_0, \dots, \Lambda_d)$. This correspondence permits to parameterize the invariant states by the density and the momentum: For each $(\rho, \boldsymbol{u})$ in $\mathfrak{A}$, let

$$\nu_{\rho,\boldsymbol{u}} = \mu_{\boldsymbol\Lambda(\rho,\boldsymbol{u})}\,.$$

2.2. Hydrodynamical Limit Under Euler Scaling

In this subsection, we deduce the hydrodynamic equation of the system under the assumption of conservation of local equilibrium. Fix smooth functions $\rho : \mathbb{R}^d \to \mathbb{R}_+$, $\boldsymbol{u} : \mathbb{R}^d \to \mathbb{R}^d$. For each $\varepsilon > 0$, denote by $\nu^\varepsilon_{\rho(\cdot),\boldsymbol{u}(\cdot)}$ the product measure on X with marginals given by

$$\nu^\varepsilon_{\rho(\cdot),\boldsymbol{u}(\cdot)}\{\eta(x,\cdot) = \xi\} = \nu_{\rho(\varepsilon x),\boldsymbol{u}(\varepsilon x)}\{\eta(0,\cdot) = \xi\}$$

for each x in $\mathbb{Z}^d$ and each configuration ξ in $\{0, 1\}^{\mathcal{V}}$. Assume that particles are initially distributed according to $\nu^\varepsilon_{\rho(\cdot),\boldsymbol{u}(\cdot)}$.

For $j = 1, \dots, d$, let ∇_j^- denote the lattice gradient acting on functions $f : \mathbb{Z}^d \to \mathbb{R}$ by $\nabla_j^- f(z) = f(z) - f(z - e_j)$. From Ito's formula, for $a = 0, \dots, d$, we have the conservation law

$$dI_a(\eta_x(t)) = \sum_{j=1}^{d} \nabla_j^- w^a_{x,j}\, dt + dM^a_x(t)\,,$$

where $M_x^a(t)$ are martingales and $w_{x,j}^a$ are the currents defined by

$$\mathcal{L}\, I_a(\eta_x) = \mathcal{L}^{ex}\, I_a(x,\eta) = \sum_{j=1}^{d} \nabla_j^- \omega_{x,j}^a\,,$$

where

$$\begin{aligned}\omega_{x,j}^0 &= \gamma\, \nabla_j^- I_0(\eta_{x+e_j}) + \sum_{v\in\mathcal{V}} (e_j\cdot v)\, b_{x,j}(v)\,,\\ \omega_{x,j}^a &= \gamma\, \nabla_j^- I_a(\eta_{x+e_j}) + \sum_{v\in\mathcal{V}} (e_a\cdot v)(e_j\cdot v) b_{x,j}(v) \quad \text{and}\\ b_{x,j}(v) &= \eta(x+e_j,v)\eta(x,v) - (1/2)[\eta(x+e_j,v)+\eta(x,v)]\,.\end{aligned}$$

In this computation, the full generator can be replaced by the exclusion operator because the collision operator preserves the density and the momentum. Let $\omega_j^a = \omega_{0,j}^a$ for $0 \leqslant a \leqslant d$, $1 \leqslant j \leqslant d$.

The expectation of the mass current in the jth direction under the local Gibbs state $\nu_{\rho(\cdot),\boldsymbol{u}(\cdot)}^{\varepsilon}$ is denoted by $\pi_{0,j}$ and given by

$$\begin{aligned}\pi_{0,j}(\rho(\varepsilon x),\boldsymbol{u}(\varepsilon x)) &:= E_{\nu_{\rho(\cdot),\boldsymbol{u}(\cdot)}^{\varepsilon}}[w_{x,j}^0]\\ &= \sum_{v\in\mathcal{V}} (v\cdot e_j)\, \theta_v(\boldsymbol{\Lambda}(\rho,\boldsymbol{u}))\, \big\{\theta_v(\boldsymbol{\Lambda}(\rho,\boldsymbol{u})) - 1\big\} + \gamma(\nabla_j^-\rho)(\varepsilon x)\,,\end{aligned}$$

while the expectation of the momentum currents $\pi_{a,j}$ are given by

$$\begin{aligned}\pi_{a,j}(\rho(\varepsilon x),\boldsymbol{u}(\varepsilon x)) &:= E_{\nu_{\rho(\cdot),\boldsymbol{u}(\cdot)}^{\varepsilon}}[w_{x,j}^a]\\ &= \sum_{v\in\mathcal{V}} (e_a\cdot v)\,(e_j\cdot v)\, \theta_v(\boldsymbol{\Lambda}(\rho,\boldsymbol{u}))\, \big\{\theta_v(\boldsymbol{\Lambda}(\rho,\boldsymbol{u})) - 1\big\} + \gamma(\nabla_j^- u_a)(\varepsilon x)\,.\end{aligned}$$

In both formulas, on the right hand side, ρ and $\boldsymbol{u}$ are evaluated at εx.

Assuming conservation of local equilibrium, it is not difficult to derive the hydrodynamic equations in the Euler scale for the lattice gas considered in this article (cf. Ref. 8). It is given by the system of hyperbolic equations

$$\begin{cases}\partial_t \rho + \sum_{j=1}^d \partial_{x_j}\pi_{0,j} = 0\,,\\ \partial_t u_a + \sum_{j=1}^d \partial_{x_j}\pi_{a,j} = 0\,.\end{cases}$$

Notice that the factors $\gamma\nabla_j^-$ do not survive in the limit due to the presence of a second derivative. They will appear, however, in the diffusive scale.

2.3. Incompressible Limit

Inserting the local equilibrium assumption into the conservation laws, on the time scale $\varepsilon^{-1}t$ one would obtain the equations

$$\begin{cases} \partial_t \rho + \varepsilon^{-1} \sum_{j=1}^d \partial_{x_j} \pi_{0,j} = \gamma \Delta \rho \,, \\ \partial_t u_a + \varepsilon^{-1} \sum_{j=1}^d \partial_{x_j} \pi_{a,j} = \gamma \Delta u_a \,. \end{cases}$$

Consider a system in which the density is near a constant and velocity is of order ε. Expanding the mass density, the momentum density and the mass and momentum currents, we obtain

$$\begin{aligned} \rho &= \rho^{(0)} + \varepsilon \rho^{(1)} + \varepsilon^2 \rho^{(2)} + \cdots \,, \\ u &= \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots \,, \\ \pi &= \pi^{(0)} + \varepsilon \pi^{(1)} + \varepsilon^2 \pi^{(2)} + \cdots \,. \end{aligned}$$

Using these expansions and assuming that the zeroth order terms of π are constants, we obtain the incompressible Navier–Stokes equations

$$\begin{cases} \sum_{j=1}^d \partial_{x_j} \pi_{a,j}^{(1)} = 0 \\ \partial_t u_a^{(1)} + \sum_{j=1}^d \partial_{x_j} \pi_{a,j}^{(2)} = \sum_{i,j} \sum_b D_{i,j}^{a,b} \partial_{x_i} \partial_{x_j} u_j^{(1)} \end{cases}$$

for $a = 0, \dots, d$ and $u_0 = \rho$. Here the diffusion coefficient $D_{i,j}^{a,b} = \gamma \delta_{a,b} \delta_{i,j}$ is a diagonal matrix. It turns out that this naive computation is correct if the diffusion coefficient is instead given by a Green–Kubo formula.

2.4. Green–Kubo Formula

For simplicity, let $\lambda_j = 0$ for $0 \leqslant j \leqslant d$ so that $\theta_v(\boldsymbol{\lambda}) = 1/2$ for every $v \in \mathcal{V}$. Denote this measure by μ_0 and let $\xi(x,v) = \eta(x,v) - \theta_v(\boldsymbol{\lambda}) = \eta(x,v) - 1/2$.

Let τ_x be the shift by x on the lattice, so that $(\tau_x \eta)(y,v) = \eta(x+y,v)$. Denote by $\langle\langle \cdot, \cdot \rangle\rangle = \langle\langle \cdot, \cdot \rangle\rangle_{\mu_0}$ the scalar product defined on X by

$$\langle\langle f, g \rangle\rangle = \sum_{x \in \mathbb{Z}^d} E_{\mu_0}[f; \tau_x g]$$

for two local functions f, g. Here $E_{\mu_0}[g;h]$ stands for the covariance between g and h: $E_{\mu_0}[g;h] = E_{\mu_0}[g\,h] - E_{\mu_0}[g] E_{\mu_0}[h]$. Let $\mathcal{G}_0$ be the space of local functions satisfying

$$E_{\mu_0}[g] = 0 \quad \text{and} \quad \langle\langle g, I_a \rangle\rangle = 0$$

for $0 \leqslant a \leqslant d$.

Let $\chi = \{\chi_{a,b},\ 0 \leqslant a, b \leqslant d\}$ be the susceptibility which in our context is given by

$$\chi_{a,b} = \langle\langle I_a, I_b \rangle\rangle = E_{\mu_0}[I_a(\eta_0); I_b(\eta_0)].$$

Denote by σ_j^a the part of the current orthogonal to the constants of motion:

$$\sigma_j^a = \omega_j^a - \sum_{b=0}^{d} c_j^{a,b} I_b(\eta_0),$$

where the coefficients $c_j^{a,b}$ are chosen for σ_j^a to belong to $\mathcal{G}_0$. An elementary computation shows that

$$c_j^{a,b} = \sum_{e=0}^{d} \langle\langle \omega_j^a, I_e \rangle\rangle (\chi^{-1})_{e,b} = \frac{\partial}{\partial \alpha_b} E_{\nu_{\boldsymbol{\alpha}}}\left[\omega_j^a\right],$$

$\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_d)$. Moreover, for $1 \leqslant a \leqslant d$,

$$\sigma_j^0 = \gamma\{I_0(\eta_{e_j}) - I_0(\eta_0)\} + \sum_{v \in \mathcal{V}} (e_j \cdot v)\xi(e_j, v)\xi(0, v),$$
$$\sigma_j^a = \gamma\{I_a(\eta_{e_j}) - I_a(\eta_0)\} + \sum_{v \in \mathcal{V}} (e_j \cdot v)(e_a \cdot v)\xi(e_j, v)\xi(0, v).$$

For simplicity assume that the susceptibility is a constant times the identity: $\chi_{a,b} = \kappa\, \delta_{a,b}$. A straightforward computation shows that this is the case if the set $\mathcal{V}$ is a cube centered at the origin. Under this assumption, following the computation presented in Section 2 of Ref. 10, we obtain that for $0 \leqslant a, b \leqslant d$, $1 \leqslant i, j \leqslant d$,

$$\begin{aligned} D_{i,j}^{a,b}(t) &:= \frac{1}{2t\kappa} \left\{ \sum_{x \in \mathbb{Z}^d} E_{\mu_0}\big[I_a(\eta_x(t)); I_b(\eta_0(0))\big] - t^2 V_{i,j}^{a,b} \right\} \\ &= \gamma\delta_{a,b}\delta_{i,j} + \frac{1}{2t\kappa} \int_0^t ds \int_0^s dr \langle\langle \sigma_i^a, e^{r\mathcal{L}} \sigma_j^b \rangle\rangle \\ &\quad + \frac{1}{2t\kappa} \int_0^t ds \int_0^s dr \langle\langle \sigma_j^a, e^{r\mathcal{L}} \sigma_i^b \rangle\rangle, \end{aligned}$$

where

$$2V_{i,j}^{a,b} = \nabla E_{\nu_\alpha}[\omega_{0,i}^a] \cdot \nabla E_{\nu_\alpha}[\omega_{0,j}^b] + \nabla E_{\nu_\alpha}[\omega_{0,i}^b] \cdot \nabla E_{\nu_\alpha}[\omega_{0,j}^a]$$

and, for a local function h, $\nabla E_{\nu_\alpha}[h] = ((\partial/\partial\alpha_0)E_{\nu_\alpha}[h], \ldots, (\partial/\partial\alpha_d)E_{\nu_\alpha}[h])$. In dimension $d \geqslant 3$, the diffusion coefficients $D_{i,j}^{a,b}(t)$ converge, as $t \uparrow \infty$, to the diffusion coefficients $D_{i,j}^{a,b}$ given by the incompressible Navier–Stokes equations in Subsection 2.3 (cf. Ref. 4).

For θ in $\mathbb{R}^d$ and r in $\mathbb{R}^{d+1}$, let

$$D_{\theta,r}(t) = \sum_{a,b=0}^{d} \sum_{i,j=1}^{d} r_a \theta_i D_{i,j}^{a,b}(t) \theta_j r_b \,.$$

We can now state the main result. Let $\mathbb{R}_*^n = \mathbb{R}^n \setminus \{0\}$.

Theorem 2.1. Fix θ in $\mathbb{R}_*^d$ and r in $\mathbb{R}_*^{d+1}$. In dimension $d = 2$, there exists a positive constant $C_0 = C_0(\theta, r)$ so that for all sufficiently small $\lambda > 0$,

$$\int_0^\infty dt\, e^{-\lambda t}\, t D_{\theta,r}(t) \geqslant C_0 \lambda^{-2} \log\log \lambda^{-1} \,.$$

Recall that $\int_0^\infty e^{-\lambda t} f(t) dt \sim \lambda^{-\alpha}$ as $\lambda \to 0$ means, in some weak sense, that $f(t) \sim t^{\alpha-1}$ as $t \to \infty$. Theorem 2.1 is therefore stating that the diffusion coefficient $D_{\theta,r}(t)$ is diverging as $\log\log t$ in a weak sense.

Fix a vector θ in $\mathbb{R}^d$, r in $\mathbb{R}^{d+1}$ and let $\sigma = \sigma_{\theta,r}$ be given by

$$\sigma = \sum_{a=0}^{d} \sum_{j=1}^{d} r_a \theta_i \sigma_i^a \,. \tag{2.3}$$

An elementary computation shows that for every $\lambda > 0$

$$\int_0^\infty dt\, e^{-\lambda t}\, t D_{\theta,r}(t) = \gamma \lambda^{-2} \|\theta\|^2 \|r\|^2 + \lambda^{-2} \langle\langle \sigma, (\lambda - \mathcal{L})^{-1} \sigma \rangle\rangle \,.$$

Therefore, Theorem 2.1 follows from next estimate on the resolvent.

Lemma 2.2. Fix θ in $\mathbb{R}_*^d$ and r in $\mathbb{R}_*^{d+1}$. There exists a positive constant $C_0 = C_0(\theta, r)$ such that for sufficiently small $\lambda > 0$,

$$\langle\langle \sigma, (\lambda - \mathcal{L})^{-1} \sigma \rangle\rangle \geqslant C_0 \log\log \lambda^{-1} \,.$$

Notice that the piece of the current $\gamma\{I_a(\eta_{e_j}) - I_a(\eta_0)\}$ vanishes for the inner product $\langle\langle \cdot, \cdot \rangle\rangle$. We may therefore ignore it in the computations below.

3. DUAL REPRESENTATION

Recall that we are fixing the chemical potentials to be zero: $\lambda_j = 0$ for $0 \leqslant j \leqslant d$ and that we denote by μ_0 the product invariant measure associated to this chemical potential. Unless otherwise stated, $\langle f, g \rangle$ stands for the inner product of f and g in $L^2(\mu_0)$.

Denote by $\mathcal{E}$ the finite subsets of $\mathbb{Z}^d \times \mathcal{V}$ and by $\mathcal{E}_n$ the subsets of $\mathcal{E}$ with cardinality n, for $n \geqslant 0$. For a set A in $\mathcal{E}$, let Ψ_A be the local function defined by

$$\Psi_A(\eta) = \prod_{(x,v)\in A} [\eta(x,v) - 1/2] = \prod_{(x,v)\in A} \xi(x,v).$$

By convention $\Psi_\phi = 1$, where ϕ stands for the empty set. It is not difficult to check that $\{\Psi_A : A \in \mathcal{E}\}$ forms an orthogonal basis of $L^2(\mu_0)$. In particular, any local function f can be written as $\sum_{A\in\mathcal{E}} \mathfrak{f}(A)\Psi_A$ for some finite supported function $\mathfrak{f}: \mathcal{E} \to \mathbb{R}$. This latter function is called the Fourier coefficients of the local function f and frequently denoted by $\mathbb{T}f$. A local function f is said to have degree $n \geqslant 0$ if $(\mathbb{T}f)(A) = 0$ for any A in $\mathcal{E}_n^c$.

Notice that for any local functions $f = \sum_{A\in\mathcal{E}} \mathfrak{f}(A)\Psi_A, g = \sum_{A\in\mathcal{E}} \mathfrak{g}(A)\Psi_A$,

$$\langle f, g \rangle_{\mu_0} = \sum_{n\geqslant 0} (1/4)^n \sum_{A\in\mathcal{E}_n} \mathfrak{f}(A)\, \mathfrak{g}(A).$$

The factor (1/4) appears because we did not consider an orthonormal basis since $E_{\mu_0}[\xi(x,v)^2] = 1/4$.

We say that two finite subsets A, B of $\mathbb{Z}^d \times \mathcal{V}$ are equivalent if one is the translation of the other. This equivalence relation is denoted by $\sim$ so that $A \sim B$ if $A = B + x$ for some x in $\mathbb{Z}^d$. Let $\tilde{\mathcal{E}}_n$ be the quotient of $\mathcal{E}_n$ with respect to this equivalence relation: $\tilde{\mathcal{E}}_n = \mathcal{E}_n/\sim$, $\tilde{\mathcal{E}} = \mathcal{E}/\sim$. An elementary computation (cf. Ref. 9) gives that

$$\langle\langle f, g \rangle\rangle_{\mu_0} = \sum_{n\geqslant 1} (1/4)^n \sum_{A\in\tilde{\mathcal{E}}_n} \bar{\mathfrak{f}}(A)\, \bar{\mathfrak{g}}(A),$$

where

$$\bar{\mathfrak{f}}(A) = \sum_{z\in\mathbb{Z}^d} \mathfrak{f}(A+z). \tag{3.1}$$

Therefore, if we denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the inner product in $L^2(\mathcal{E})$ defined by

$$\langle\langle \mathfrak{f}, \mathfrak{g} \rangle\rangle = \sum_{n \geqslant 1} (1/4)^n \sum_{A \in \tilde{\mathcal{E}}_n} \bar{\mathfrak{f}}(A)\, \bar{\mathfrak{g}}(A)\,,$$

where $\bar{\mathfrak{f}}$, $\bar{\mathfrak{g}}$ are defined by (3.1), we have that

$$\langle\langle f, g \rangle\rangle_{\mu_0} = \langle\langle \mathbb{T} f, \mathbb{T} g \rangle\rangle\,.$$

The goal of this section is to examine the action of the generators $\mathcal{L}^{ex}$, $\mathcal{L}^c$ on the Fourier coefficients. More precisely, to find operators $\mathbb{L}^{ex}$, $\mathbb{L}^c$ such that $\mathbb{L}^{ex}\mathbb{T} = \mathbb{T}\mathcal{L}^{ex}$, $\mathbb{L}^c\mathbb{T} = \mathbb{T}\mathcal{L}^c$.

3.1. The Exclusion Operator

We start with the exclusion part of the generator which can be decomposed into its symmetric part $\mathcal{S}$ and its antisymmetric part $\mathcal{A}$ as given by

$$(\mathcal{S} f)(\eta) = \gamma \sum_{v \in \mathcal{V}} \sum_{x \in \mathbb{Z}^d} \sum_{j=1}^{d} \{ f(\eta^{x, x+e_j, v}) - f(\eta) \}\,,$$

$$(\mathcal{A} f)(\eta) = \frac{1}{2} \sum_{v \in \mathcal{V}} \sum_{j=1}^{d} (e_j \cdot v) \sum_{x \in \mathbb{Z}^d} \{ \eta(x, v) - \eta(x+e_j, v) \} \{ f(\eta^{x, x+e_j, v}) - f(\eta) \}\,.$$

To examine the action of the symmetric part of the exclusion generator on the Fourier coefficients, we first compute $\mathcal{S}\Psi_A$. An elementary computation shows that for each set A in $\mathcal{E}$,

$$\mathcal{S}\Psi_A = \gamma \sum_{v \in \mathcal{V}} \sum_{j=1}^{d} \sum_{x \in \mathbb{Z}^d} \{ \Psi_{A_{x, x+e_j, v}} - \Psi_A \}$$

provided $A_{x, x+e_j, v}$ stands for

$$\begin{cases} (A \setminus \{(x, v)\}) \cup \{(x+e_j, v)\} & \text{if } (x, v) \in A \text{ and } (x+e_j, v) \notin A, \\ (A \setminus \{(x+e_j, v)\}) \cup \{(x, v)\} & \text{if } (x+e_j, v) \in A \text{ and } (x, v) \notin A, \\ A & \text{otherwise.} \end{cases} \tag{3.2}$$

In particular, for any local function $f = \sum_A \mathfrak{f}(A)\Psi_A$, a change of variable $B = A_{x,x+e_j,v}$ gives that

$$\mathcal{S}f = \gamma \sum_{A\in\mathcal{E}} \Psi_A \sum_{v\in\mathcal{V}} \sum_{x\in\mathbb{Z}^d} \sum_{j=1}^{d} \{\mathfrak{f}(A_{x,x+e_j,v}) - \mathfrak{f}(A)\}\,.$$

Therefore, if we define the operator $\mathbb{S}$ as

$$(\mathbb{S}\mathfrak{f})(A) = \gamma \sum_{v\in\mathcal{V}} \sum_{x\in\mathbb{Z}^d} \sum_{j=1}^{d} \{\mathfrak{f}(A_{x,x+e_j,v}) - \mathfrak{f}(A)\}\,,$$

we have that $\mathbb{T}\mathcal{S} = \mathbb{S}\mathbb{T}$.

We turn now to the antisymmetric part. To compute $\mathcal{A}\Psi_A(\eta)$, observe that $[\eta(y,w) - 1/2]\Psi_A$ is equal to $\Psi_{A\cup\{(y,w)\}}$ if (y,w) does not belong to A and is equal to $(1/4)\Psi_{A\setminus\{(y,w)\}}$ if (y,w) belongs to A because $(\eta(y,v) - 1/2)^2 = 1/4$. In particular, a straightforward computation shows that

$$\begin{aligned}\mathcal{A}\Psi_A = &\sum_{j=1}^{d} \sum_{\substack{(x,v)\in A\\(x+e_j,v)\notin A}} (e_j\cdot v)\{\Psi_{A\cup\{(x+e_j,v)\}} - (1/4)\Psi_{A\setminus\{(x,v)\}}\}\\ &+\sum_{j=1}^{d} \sum_{\substack{(x+e_j,v)\in A\\(x,v)\notin A}} (e_j\cdot v)\{(1/4)\Psi_{A\setminus\{(x+e_j,v)\}} - \Psi_{A\cup\{(x,v)\}}\}\,.\end{aligned}$$

In the first term on the right hand side, the second sum is carried over all pairs (x,v) in $\mathbb{Z}^d\times\mathcal{V}$, such that (x,v) belongs to A and $(x+e_j,v)$ does not. Therefore, if $f = \sum_{A\in\mathcal{E}} \mathfrak{f}(A)\Psi_A$ is a local function, after elementary changes of variables, we obtain that

$$\mathcal{A}f = \sum_{A\in\mathcal{E}} (\mathbb{A}\mathfrak{f})(A)\Psi_A$$

provided $\mathbb{A} = \mathbb{J}_+ + \mathbb{J}_-$, where

$$(\mathbb{J}_+\mathfrak{f})(A) = \sum_{j=1}^{d} \sum_{\substack{(x,v)\in A\\(x+e_j,v)\in A}} (e_j\cdot v)\big\{\mathfrak{f}(A\setminus\{(x+e_j,v)\}) - \mathfrak{f}(A\setminus\{(x,v)\})\big\}\,,$$

$$(\mathbb{J}_{-}\mathfrak{f})(A) = (1/4)\sum_{j=1}^{d}\sum_{\substack{(x,v)\notin A\\(x+e_j,v)\notin A}}(e_j\cdot v)\big\{\mathfrak{f}(A\cup\{(x+e_j,v)\})-\mathfrak{f}(A\cup\{(x,v)\})\big\}$$

and $\mathbb{T}\mathcal{A}=\mathbb{A}\mathbb{T}$.

Let $\mathbb{L}^{ex}=\mathbb{S}+\mathbb{A}$. Up to this point we proved that $\mathbb{T}\mathcal{L}^{ex}=\mathbb{L}^{ex}\mathbb{T}$. This operator $\mathbb{L}^{ex}$, which is not a generator, can thus be decomposed in three pieces, $\mathbb{S}$, $\mathbb{J}_{+}$, $\mathbb{J}_{-}$. $\mathbb{S}$ is the symmetric part of $\mathbb{L}^{ex}$ and does not change the degree of a function. In contrast, $\mathbb{J}_{+}$ increases the degree by one, while $\mathbb{J}_{-}$ decreases it by one and $\mathbb{J}_{-}$ is the adjoint of $-\mathbb{J}_{+}$: $(\mathbb{J}_{-})^{*}=-\mathbb{J}_{+}$. In particular, $\mathbb{J}_{-}+\mathbb{J}_{+}$ is anti-symmetric:

$$\begin{gathered}\langle\langle\mathbb{S}\mathfrak{f},\mathfrak{g}\rangle\rangle=\langle\langle\mathfrak{f},\mathbb{S}\mathfrak{g}\rangle\rangle \quad\text{and}\quad \langle\langle\mathbb{J}_{+}\mathfrak{f},\mathfrak{g}\rangle\rangle=-\langle\langle\mathfrak{f},\mathbb{J}_{-}\mathfrak{g}\rangle\rangle\\ \text{so that}\quad \langle\langle(\mathbb{J}_{+}+\mathbb{J}_{-})\mathfrak{f},\mathfrak{g}\rangle\rangle=-\langle\langle\mathfrak{f},(\mathbb{J}_{+}+\mathbb{J}_{-})\mathfrak{g}\rangle\rangle\,.\end{gathered}$$

Moreover, a simple computation shows that in $L^2(\mathcal{E})$

$$\begin{aligned}\mathbb{S}\mathfrak{f}&=0\quad\text{for all functions }\mathfrak{f}\text{ of degree one},\\ \mathbb{J}_{-}\mathfrak{f}&=0\quad\text{for all functions }\mathfrak{f}\text{ of degree two}.\end{aligned}\tag{3.3}$$

3.2. The Collision Operator

The remainder of this section is devoted to the collision operator. We start defining a generator $\mathcal{L}_1^c$ and showing in Lemma 3.1 below that it is of the same order as $\mathcal{L}^c$ for our purposes. We conclude the section investigating the action of $\mathcal{L}_1^c$ on the Fourier coefficients.

Fix a site x in $\mathbb{Z}^d$ and a point $q=(v,w,v',w')$ in the collision set $\mathcal{Q}$. Let $q'=(v,w,w',v')$. Denote by $\mathcal{L}_{x,q}$ the generator defined by

$$(\mathcal{L}_{x,q}f)(\eta)=p'(x,q,\eta)\{[f(\eta^{x,q})-f(\eta)]+[f(\eta^{x,q'})-f(\eta)]\},$$

where

$$\begin{aligned}p'(x,q,\eta)=&\;\eta(x,v)\,\eta(x,w)\,[1-\eta(x,v')][1-\eta(x,w')]\\ &+\eta(x,v')\,\eta(x,w')\,[1-\eta(x,v)][1-\eta(x,w)].\end{aligned}$$

Since the collision set $\mathcal{Q}$ is symmetric, we may rewrite the collision generator $\mathcal{L}^c$ as

$$\mathcal{L}^c=(1/4)\sum_{x\in\mathbb{Z}^d}\sum_{q\in\mathcal{Q}}\mathcal{L}_{x,q}\,.$$

Fix a site x in $\mathbb{Z}^d$. We start the analysis of the collision operator by examining the generator $\mathcal{L}_{x,q}$. Since site x is fixed, we omit the index x below so that $\mathcal{L}_q = \mathcal{L}_{x,q}$. We also denote by ζ configurations of $\{0,1\}^{\mathcal{V}}$ and by $\xi(v)$ the function $\zeta(v) - 1/2$. Since m_λ given by (2.1) is a product measure on $\{0,1\}^{\mathcal{V}}$, $\{\prod_{v\in B} \xi(v): B \subset V\}$ forms an orthogonal basis of $L^2(m_0)$ if $\prod_{v\in\phi} \xi(v) = 1$, where ϕ stands for the empty set.

Fix $q = (v, w, v', w')$ in $\mathcal{Q}$ and let $H_q = \{v, w, v', w'\}$. Since $\mathcal{L}_q$ only changes the variables $\{\zeta(u): u \in H_q\}$, all other variables can be considered as constants so that for any subset B of $\mathcal{V}$,

$$\mathcal{L}_q \prod_{u\in B} \zeta(u) = \prod_{u\in B\setminus H_q} \zeta(u)\, \mathcal{L}_q \prod_{u\in B\cap H_q} \zeta(u)\,.$$

A similar identity holds if we replace ζ by ξ.

Let

$$\begin{aligned}
\phi_1(\xi) &= \xi(v') + \xi(w') - \xi(v) - \xi(w)\,,\\
\tilde{\phi}_3(\xi) &= \xi(v)\xi(w)\xi(v') + \xi(v)\xi(w)\xi(w') - \xi(v')\xi(w')\xi(v) - \xi(v')\xi(w')\xi(w)\,.
\end{aligned}$$

The index 1 and 3 stand for the degree of the functions involved. Straightforward computations give the following identities for degree one functions:

$$\begin{aligned}
\mathcal{L}_q\xi(v) = \mathcal{L}_q\xi(w) &= (1/2)\phi_1(\xi) + 2\tilde{\phi}_3(\xi)\,,\\
\mathcal{L}_q\xi(v') = \mathcal{L}_q\xi(w') &= -(1/2)\phi_1(\xi) - 2\tilde{\phi}_3(\xi)\,.
\end{aligned}$$

Degree two functions vanish under the action of the generator:

$$\mathcal{L}_q\xi(u_1)\xi(u_2) = 0$$

for u_1, u_2 in H_q. To derive these identities we used that $\mathcal{L}_q\{\xi(v) + \xi(v')\} = 0$ and similar equalities. Degree three functions are such that

$$\begin{aligned}
\mathcal{L}_q\xi(v)\xi(w)\xi(v') = \mathcal{L}_q\xi(v)\xi(w)\xi(w') &= -(1/8)\phi_1(\xi) - (1/2)\tilde{\phi}_3(\xi)\,,\\
\mathcal{L}_q\xi(v')\xi(w')\xi(v) = \mathcal{L}_q\xi(v')\xi(w')\xi(w) &= (1/8)\phi_1(\xi) + (1/2)\tilde{\phi}_3(\xi)\,.
\end{aligned}$$

Finally, degree four functions vanish under the action of the generator:

$$\mathcal{L}_q\xi(v)\xi(w)\xi(v')\xi(w') = 0\,.$$

Here again, to deduce this equality we used that $\mathcal{L}_q\{\xi(v)\xi(w)\xi(v') + \xi(v')\xi(w')\xi(v)\}$ vanishes as well as similar identities.

It follows from the previous formulas that the unique non zero eigenvalue -4 is associated to the eigenfunction $\psi = \phi_1 + 4\tilde{\phi}_3$. In particular, the generator $\mathcal{L}_q$ can be written as

$$\mathcal{L}_q f = -4 \frac{\langle f, \psi\rangle}{\langle \psi, \psi\rangle} \psi \,,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(m_0)$. Denote by $\mathcal{L}_{q,1}$, $\mathcal{L}_{q,3}$ the operators defined by

$$\mathcal{L}_{q,1} f = -4 \frac{\langle f, \phi_1\rangle}{\langle \phi_1, \phi_1\rangle} \phi_1 \,, \quad \mathcal{L}_{q,3} f = -4 \frac{\langle f, \phi_3\rangle}{\langle \phi_3, \phi_3\rangle} \phi_3$$

for $\phi_3 = 4\tilde{\phi}_3$. Since ϕ_1, ϕ_3 are orthogonal, an elementary computation shows that

$$-\mathcal{L}_q \leqslant -2\mathcal{L}_{q,1} - 2\mathcal{L}_{q,3} \tag{3.4}$$

in the matrix sense.

If we reintroduce the index x, we obtain the operators

$$\mathcal{L}^c_j = (1/4) \sum_{x\in\mathbb{Z}^d} \sum_{q\in Q} \mathcal{L}_{x,q,j}$$

for $j = 1$, 3. Notice that both operators keep the degree of local functions. Indeed, for a local function f, an elementary computation shows that

$$\mathcal{L}^c_j f = - \sum_{x\in\mathbb{Z}^d} \sum_{q\in Q} \langle f, \phi_{x,q,j}\rangle_x \phi_{x,q,j}$$

because $\langle \phi_{x,q,j}, \phi_{x,q,j}\rangle_x = 1$ for all x, q and j. In this formula $\langle \cdot, \cdot\rangle_x$ stands for the inner product with respect to $m_0(\eta(x,\cdot))$, which means that only the variables at site x are integrated and $\phi_{x,q,j}$ are the functions defined by

$$\begin{aligned}
\phi_{x,q,1}(\eta) &= \xi(x,v') + \xi(x,w') - \xi(x,v) - \xi(x,w)\,,\\
\phi_{x,q,3}(\eta) &= 4\xi(x,v)\xi(x,w)\xi(x,v') + 4\xi(x,v)\xi(x,w)\xi(x,w')\\
&\quad -4\xi(x,v')\xi(x,w')\xi(x,v) - 4\xi(x,v')\xi(x,w')\xi(x,w)\,.
\end{aligned}$$

If $f=\Psi_B$ for some finite set B, an elementary computation shows that

$$\langle \Psi_B, \phi_{x,q,j}\rangle_x \phi_{x,q,j}$$

has the same degree as Ψ_B for all x, q and j, which proves the claim.

It follows from (3.4) that

$$-\mathcal{L}^c \leqslant -2\mathcal{L}_1^c - 2\mathcal{L}_3^c .$$

In order to have a tractable algebraic expression for the collision operator, we plan to substitute $\mathcal{L}^c$ by $\mathcal{L}_1^c$. In order to estimate the third degree terms we use the following lemma.

Lemma 3.1. There exists a finite constant C_0 such that

$$\begin{aligned} & {C_0}^{-1} \langle\langle f, \{\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c\}^{-1} f\rangle\rangle \\ & \quad \leqslant \langle\langle f, \{\lambda - \mathcal{L}\}^{-1} f\rangle\rangle \leqslant C_0 \langle\langle f, \{\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c\}^{-1} f\rangle\rangle \end{aligned}$$

for every $\lambda > 0$ and every local function f.

The proof of this result is based on the lemma below whose proof is similar to the one of Lemma 4.2 in Ref. 11.

Lemma 3.2. Consider a function $\omega\colon \mathcal{E}_3 \to \mathbb{R}$. Assume that there exists $\ell \geqslant 1$ such that $\omega((x_1, v_1), (x_2, v_2), (x_3, v_3)) = 0$ if $|x_1 - x_2| > \ell$ or $|x_1 - x_3| > \ell$. Then, there exists a finite constant C_0 depending only on ω such that

$$\sum_{(\boldsymbol{x},\boldsymbol{v})\in\mathcal{E}_3} \mathfrak{f}^2(\boldsymbol{x},\boldsymbol{v})\,\omega(\boldsymbol{x},\boldsymbol{v}) \leqslant C_0 \sum_{(\boldsymbol{x},\boldsymbol{v})\in\mathcal{E}_3} \mathfrak{f}(\boldsymbol{x},\boldsymbol{v})(-\mathbb{S}\mathfrak{f})(\boldsymbol{x},\boldsymbol{v})$$

for every finite supported function $\mathfrak{f}\colon \mathcal{E}_3 \to \mathbb{R}$.

Proof of Lemma 3.1. We claim that $-\mathcal{L}_3^c \leqslant -C_0\mathcal{S}$ for some finite constant C_0. Since $\mathcal{L}_3^c$ keeps the degree, to prove the claim we only need to show that

$$\langle -\mathcal{L}_3^c f, f\rangle \leqslant C_0 \langle -\mathcal{S} f, f\rangle$$

for local functions of a fixed degree.

Fix $n \geqslant 1$ and a local function f of degree n. By definition of $\mathcal{L}_3^c$, taking conditional expectations we obtain that

$$\langle -\mathcal{L}_3^c f, f \rangle = \sum_{q \in \mathcal{Q}} \sum_{x \in \mathbb{Z}^d} E\Big[\langle f, \phi_{x,q,3} \rangle_x^2 \Big].$$

Since the velocity set is finite, by definition of $\phi_{x,q,3}$, to prove the claim it is enough to show that

$$\sum_{x \in \mathbb{Z}^d} E\Big[\langle f, \Psi_{B_x} \rangle_x^2 \Big] \leqslant C_0 \langle (-\mathcal{S}) f, f \rangle ,$$

where $B_x = \{(x, v_1), (x, v_2), (x, v_3)\}$ and v_1, v_2, v_3 are three distinct velocities in $\mathcal{V}$. Assume that $f = \sum_B \mathfrak{f}(B) \Psi_B$. An elementary computation shows that the expectation appearing on the left hand side of the previous inequality is bounded above by

$$(1/4)^n \sum_{B \supset B_x} \mathfrak{f}(B)^2 ,$$

where the sum is performed over all sets B which contain B_x. In particular, if for a finite set $A = \{(x_1, v_1), (x_2, v_2), (x_3, v_3)\}$, we set

$$\rho(A)^2 = (1/4)^n \sum_{B \supset A} \mathfrak{f}(B)^2 ,$$

where the summation is performed over all sets B which contain A, we just proved that

$$\langle -\mathcal{L}_3^c f, f \rangle \leqslant \sum_{v_1, v_2, v_3} \sum_{x \in \mathbb{Z}^d} \rho(\{(x, v_1), (x, v_2), (x, v_3)\})^2 .$$

By Lemma 3.2, this expression is less than or equal to

$$C_0 \sum_{(\boldsymbol{x}, \boldsymbol{v}) \in \mathcal{E}_3} \rho(\boldsymbol{x}, \boldsymbol{v}) (-\mathbb{S} \rho)(\boldsymbol{x}, \boldsymbol{v}) \leqslant C_0 \langle -\mathcal{S} f, f \rangle .$$

Last inequality follows from Schwarz inequality and concludes the proof of the claim. Here and below, C_0, a_0 are constants whose value may change from line to line.

Since $-\mathcal{L}_3^c \leqslant -C_0\mathcal{S}$, we have that $-\mathcal{L}^c \leqslant -2(\mathcal{L}_1^c + \mathcal{L}_3^c) \leqslant -C_0(\mathcal{S} + \mathcal{L}_1^c)$ and

$$\lambda - \mathcal{S} - \mathcal{L}^c \leqslant a_0\{\lambda - \mathcal{S} - \mathcal{L}_1^c\}$$

for some finite constant $a_0 > 1$ and all $\lambda > 0$.

On the other hand, since $\mathcal{L}_3^c \geqslant C_0\mathcal{S}$ and since $(a-b)^2 \geqslant (1-\varepsilon)a^2 - (\varepsilon^{-1}-1)b^2$ for every $0 < \varepsilon < 1$, a straightforward computation shows that

$$-\mathcal{L}^c \geqslant -\frac{(1-\varepsilon)}{2}\mathcal{L}_1^c + \frac{(\varepsilon^{-1}-1)}{2}\mathcal{L}_3^c \geqslant -\frac{(1-\varepsilon)}{2}\mathcal{L}_1^c + C_0(\varepsilon^{-1}-1)\mathcal{S}$$

for some finite constant C_0. Here the factor 1/2 appeared because $\langle\psi, \psi\rangle = 2\ \langle\phi_1, \phi_1\rangle = 2\langle\phi_3, \phi_3\rangle$. If we choose ε small enough for $C_0(\varepsilon^{-1}-1) < 1$, it follows from this inequality that

$$\begin{aligned}\lambda - \mathcal{S} - \mathcal{L}^c &\geqslant \lambda - \frac{(1-\varepsilon)}{2}\mathcal{L}_1^c - \big\{1 - C_0(\varepsilon^{-1}-1)\big\}\mathcal{S}\\ &\geqslant a_0^{-1}\big\{\lambda - \mathcal{S} - \mathcal{L}_1^c\big\}\end{aligned}$$

for some finite constant $a_0 > 1$ and all $\lambda > 0$.

Up to this point we proved the existence of a finite constant $a_0 > 1$ such that

$$a_0^{-1}\big\{\lambda - \mathcal{L}_1^c - \mathcal{S}\big\} \leqslant \lambda - \mathcal{S} - \mathcal{L}^c \leqslant a_0\{\lambda - \mathcal{S} - \mathcal{L}_1^c\} \tag{3.5}$$

for all $\lambda > 0$.

It remains to add the asymmetric part of the exclusion generator. Denote by R^s (resp. R^a, R^*) the symmetric part (resp. asymmetric part, adjoint) of an operator R. It is well known that

$$\big\{(R^{-1})^s\big\}^{-1} = R^*(R^s)^{-1}R = R^s + (R^a)^*(R^s)^{-1}R^a\,.$$

In particular, for every $\lambda > 0$,

$$\Big(\big\{(\lambda - \mathcal{L})^{-1}\big\}^s\Big)^{-1} = (\lambda - \mathcal{S} - \mathcal{L}^c) + \mathcal{A}^*(\lambda - \mathcal{S} - \mathcal{L}^c)^{-1}\mathcal{A}\,.$$

In view of (3.5), there exists a finite constant $a_0 > 1$ such that

$$\begin{aligned} a_0^{-1}\Big\{(\lambda - \mathcal{S} - \mathcal{L}_1^c) + \mathcal{A}^*(\lambda - \mathcal{S} - \mathcal{L}_1^c)^{-1}\mathcal{A}\Big\} \\ \leqslant (\lambda - \mathcal{S} - \mathcal{L}^c) + \mathcal{A}^*(\lambda - \mathcal{S} - \mathcal{L}^c)^{-1}\mathcal{A} \\ \leqslant a_0\Big\{(\lambda - \mathcal{S} - \mathcal{L}_1^c) + \mathcal{A}^*(\lambda - \mathcal{S} - \mathcal{L}_1^c)^{-1}\mathcal{A}\Big\} \end{aligned}$$

so that

$$a_0^{-1}\big\{(\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\big\}^s \leqslant \big\{(\lambda - \mathcal{L})^{-1}\big\}^s \leqslant a_0\big\{(\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\big\}^s ,$$

which proves the lemma in the case of the inner product of $L^2(\mu_0)$. The extension to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ is standard (cf. Ref. 12). ∎

We conclude the section examining the action of $\mathcal{L}_1^c$ on the Fourier coefficients. For any local function $f = \sum_{A\in\mathcal{E}} \mathfrak{f}(A)\Psi_A$, a simple computation shows that

$$\mathcal{L}_1^c f = \sum_{A\in\mathcal{E}} (\mathbb{L}_1^c \mathfrak{f})(A)\Psi_A ,$$

where

$$\begin{aligned} (\mathbb{L}_1^c\mathfrak{f})(A) \;=\; (1/4)\sum_{q\in\mathcal{Q}}\sum_{x} i_q(A_x)\big\{\mathfrak{f}(A_x^c \cup \{(x,v')\}) + \mathfrak{f}(A_x^c \cup \{(x,w')\}) \\ - \mathfrak{f}(A_x^c \cup \{(x,v)\}) - \mathfrak{f}(A_x^c \cup \{(x,w)\})\big\} . \end{aligned}$$

In this formula, $q = (v, w, v', w')$, A_x stands for the set of velocities u such that (x, u) belongs to A: $A_x = \{u \in \mathcal{V}: (x, u) \in A\}$, A_x^c for the set of points (y, u) in A with $y \neq x$: $A_x^c = \{(y, u) \in A: y \neq x\}$ and if $i_q(A_x) = 1$ if A_x is an incoming velocity, -1 is A_x is an outgoing velocity and 0 otherwise:

$$i_q(A_x) = \mathbf{1}\{A_x = \{v\}\} + \mathbf{1}\{A_x = \{w\}\} - \mathbf{1}\{A_x = \{v'\}\} - \mathbf{1}\{A_x = \{w'\}\} .$$

With this notation we have that $\mathbb{T}\mathcal{L}_1^c = \mathbb{L}_1^c\mathbb{T}$. Notice that $-\mathbb{L}_1^c$ is a non-negative symmetric operator in $L^2(\mathcal{E}_n)$ and that

$$\langle -\mathbb{L}_1^c\mathfrak{f}, \mathfrak{f}\rangle = \sum_x \sum_q E_{\mu_0}\Big[\langle f, \phi_{x,q,1}\rangle_x^2\Big] .$$

4. CUTOFF OF LARGE DEGREES

For $n \geqslant 1$, let $\mathcal{G}_n = \cup_{1 \leqslant k \leqslant n} \mathcal{E}_n$. Denote by π_n the orthogonal projection on $L^2(\mathcal{E}_n)$, by P_n the orthogonal projection on $L^2(\mathcal{G}_n)$ and by $\mathcal{L}_n$ the operator $\mathcal{L}^{ex} + \mathcal{L}_1^c$ truncated at level n: $\mathcal{L}_n = P_n(\mathcal{L}^{ex} + \mathcal{L}_1^c) P_n$. In particular, $\mathfrak{f} = \sum_{n \geqslant 1} \pi_n \mathfrak{f}$ and $P_n = \sum_{1 \leqslant j \leqslant n} \pi_j$.

To investigate the asymptotic behavior of $\langle\langle \sigma, (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1} \sigma \rangle\rangle$, for $\lambda > 0$ consider the resolvent equation $(\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c) u_\lambda = \sigma$. In the Fourier space, the equation becomes the hierarchy equations

$$\begin{cases} \mathbb{J}_+^* \pi_3 \mathfrak{u}_\lambda + (\lambda - \mathbb{S} - \mathbb{L}_1^c) \pi_2 \mathfrak{u}_\lambda = \sigma \,, \\ \mathbb{J}_+^* \pi_{k+1} \mathfrak{u}_\lambda + (\lambda - \mathbb{S} - \mathbb{L}_1^c) \pi_k \mathfrak{u}_\lambda - \mathbb{J}_+ \pi_{k-1} \mathfrak{u}_\lambda = 0 \,, \quad \text{for } k \geqslant 3 \end{cases}$$

because $\mathbb{J}_+^* = -\mathbb{J}_-$ and because σ has degree 2. The hierarchy starts at degree 2 instead of 1 because the degree one equation is trivial. Indeed, by (3.3), $(\mathbb{S} + \mathbb{L}_1^c) \pi_1 \mathfrak{u}_\lambda = 0$, $\mathbb{J}_- \pi_2 \mathfrak{u}_\lambda = 0$, so that the degree one equation

$$-\mathbb{J}_- \pi_2 \mathfrak{u}_\lambda + (\lambda - \mathbb{S} - \mathbb{L}_1^c) \pi_1 \mathfrak{u}_\lambda = 0$$

becomes $\pi_1 \mathfrak{u}_\lambda = 0$. Hence $\pi_1 \mathfrak{u}_\lambda$ plays no role and we can set $\pi_1 \mathfrak{u}_\lambda = 0$.

Notice that we are using the same notation σ for the local function defined in (2.3) and its Fourier transform $\sigma : \mathcal{E} \to \mathbb{R}$ which is given by

$$\sigma(A) = \theta_j \left\{ r_0 (e_j \cdot v) + \sum_{a=1}^{d} r_a (e_a \cdot v)(e_j \cdot v) \right\} \tag{4.1}$$

if $A = \{(0, v), (e_j, v)\}$ for some v in $\mathcal{V}$, $1 \leqslant j \leqslant d$, and $\sigma(A) = 0$, otherwise.

Consider the truncated resolvent equation up to the degree n:

$$\begin{cases} \mathbb{J}_+^* \pi_3 \mathfrak{u}_{\lambda,n} + (\lambda - \mathbb{S} - \mathbb{L}_1^c) \pi_2 \mathfrak{u}_{\lambda,n} = \sigma \,, \\ \mathbb{J}_+^* \pi_{k+1} \mathfrak{u}_{\lambda,n} + (\lambda - \mathbb{S} - \mathbb{L}_1^c) \pi_k \mathfrak{u}_{\lambda,n} - \mathbb{J}_+ \pi_{k-1} \mathfrak{u}_{\lambda,n} = 0 \,, \quad 3 \leqslant k \leqslant n-1 \,, \\ (\lambda - \mathbb{S} - \mathbb{L}_1^c) \pi_n \mathfrak{u}_{\lambda,n} - \mathbb{J}_+ \pi_{n-1} \mathfrak{u}_{\lambda,n} = 0 \,. \end{cases} \tag{4.2}$$

We can solve the final equation of (4.2) by

$$\pi_n \mathfrak{u}_{\lambda,n} = (\lambda - \mathbb{S} - \mathbb{L}_1^c)^{-1} \mathbb{J}_+ \pi_{n-1} \mathfrak{u}_{\lambda,n} \,.$$

Substituting this into the equation of degree $n-1$, we have

$$\pi_{n-1}\mathfrak{u}_{\lambda,n} = \left[(\lambda - \mathbb{S} - \mathbb{L}_1^c) + \mathbb{J}_+^*(\lambda - \mathbb{S} - \mathbb{L}_1^c)^{-1}\mathbb{J}_+\right]^{-1} \mathbb{J}_+\pi_{n-2}\mathfrak{u}_{\lambda,n}\,.$$

Solving iteratively we arrive at

$$\pi_2\mathfrak{u}_{\lambda,n} = \mathcal{T}_n\sigma\,,$$

where the operators $\{\mathcal{T}_n, n \geqslant 2\}$ are defined inductively by

$$\mathcal{T}_2 = (\lambda - \mathbb{S} - \mathbb{L}_1^c)^{-1}\,, \quad \mathcal{T}_{n+1} = \left\{(\lambda - \mathbb{S} - \mathbb{L}_1^c) + \mathbb{J}_+^*\mathcal{T}_n^{-1}\mathbb{J}_+\right\}^{-1}. \tag{4.3}$$

The truncated equation represents the solution of $(\lambda - \mathcal{L}_n)\mathfrak{u}_{\lambda,n} = \sigma$ and hence $\langle\langle \pi_2\mathfrak{u}_{\lambda,n}, \sigma\rangle\rangle = \langle\langle \sigma, (\lambda - \mathcal{L}_n)^{-1}\sigma\rangle\rangle$ so that

$$\langle\langle \sigma, (\lambda - \mathcal{L}_n)^{-1}\sigma\rangle\rangle = \langle\langle \sigma, \mathcal{T}_n\sigma\rangle\rangle\,,$$

where, for example,

$$\begin{aligned}
\mathcal{T}_3 &= \left\{(\lambda - \mathbb{S} - \mathbb{L}_1^c) + \mathbb{J}_-(\lambda - \mathbb{S} - \mathbb{L}_1^c)^{-1}\mathbb{J}_+\right\}^{-1}, \\
\mathcal{T}_4 &= \left[(\lambda - \mathbb{S} - \mathbb{L}_1^c) + \mathbb{J}_-\left\{(\lambda - \mathbb{S} - \mathbb{L}_1^c) + \mathbb{J}_-(\lambda - \mathbb{S} - \mathbb{L}_1^c)^{-1}\mathbb{J}_+\right\}^{-1}\mathbb{J}_+\right]^{-1}.
\end{aligned}$$

Lemma 4.1. For each $\lambda > 0$, $\langle\langle \sigma, (\lambda - \mathcal{L}_{2k+1})^{-1}\sigma\rangle\rangle$ is an increasing sequence which converges to $\langle\langle \sigma, (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\sigma\rangle\rangle$ and $\langle\langle \sigma, (\lambda - \mathcal{L}_{2k})^{-1}\sigma\rangle\rangle$ is a decreasing sequence which converges to $\langle\langle \sigma, (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\sigma\rangle\rangle$.

Proof. Since $\lambda - \mathbb{S} - \mathbb{L}_1^c$ is positive, it is easy to show from the definition of the sequence of operators $\mathcal{T}_n$ that $0 \leqslant \mathcal{T}_3 \leqslant \mathcal{T}_2$ and that $\mathcal{T}_m \leqslant \mathcal{T}_n$ if $\mathcal{T}_{m-1} \geqslant \mathcal{T}_{n-1}$. In particular, $\{\mathcal{T}_{2k}, k \geqslant 1\}$ is a decreasing sequence, $\{\mathcal{T}_{2k+1}, k \geqslant 1\}$ is an increasing sequence and $\mathcal{T}_{2k+1} \leqslant \mathcal{T}_{2j}$ for any $k, j \geqslant 1$:

$$\begin{aligned}
&\langle\langle \sigma, (\lambda - \mathcal{L}_3)^{-1}\sigma\rangle\rangle \leqslant \langle\langle \sigma, (\lambda - \mathcal{L}_5)^{-1}\sigma\rangle\rangle \leqslant \cdots \\
&\quad \cdots \leqslant \langle\langle \sigma, (\lambda - \mathcal{L}_4)^{-1}\sigma\rangle\rangle \leqslant \langle\langle \sigma, (\lambda - \mathcal{L}_2)^{-1}\sigma\rangle\rangle\,.
\end{aligned} \tag{4.4}$$

To check that $\langle\langle \sigma, (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\sigma\rangle\rangle$ is in fact the limit of these upper and lower bounds we use the variational formula. For any matrix

M, let M_s denote the symmetric part $(M+M^*)/2$. The identity $\left\{[M^{-1}]_s\right\}^{-1} = M^*(M_s)^{-1}M$ always holds, and thus we have

$$\begin{aligned}
&\langle\langle \sigma, (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\sigma\rangle\rangle \\
&\quad = \sup_{\mathfrak{f}} \left\{ 2\langle\langle \sigma, \mathfrak{f}\rangle\rangle - \langle\langle (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)\mathfrak{f}, (\lambda - \mathbb{S} - \mathbb{L}_1^c)^{-1}(\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)\mathfrak{f}\rangle\rangle \right\},
\end{aligned} \tag{4.5}$$

where the supremum is carried over all finite supported functions $\mathfrak{f}: \mathcal{E} \to \mathbb{R}$. Note that

$$\begin{aligned}
&\langle\langle (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)\mathfrak{f}, (\lambda - \mathbb{S} - \mathbb{L}_1^c)^{-1}(\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)\mathfrak{f}\rangle\rangle \\
&\quad = \langle\langle \mathfrak{f}, (\lambda - \mathbb{S} - \mathbb{L}_1^c)\mathfrak{f}\rangle\rangle + \langle\langle \mathbb{A}\mathfrak{f}, (\lambda - \mathbb{S} - \mathbb{L}_1^c)^{-1}\mathbb{A}\mathfrak{f}\rangle\rangle,
\end{aligned}$$

where $\mathbb{A} = \mathbb{J}_+ + \mathbb{J}_-$. Hence,

$$\begin{aligned}
\langle\langle \sigma, (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\sigma\rangle\rangle = \sup_{\mathfrak{f}} \inf_{\mathfrak{g}} \Big\{ & 2\langle\langle \sigma - \mathbb{A}^*\mathfrak{g}, \mathfrak{f}\rangle\rangle \\
& - \langle\langle \mathfrak{f}, (\lambda - \mathbb{S} - \mathbb{L}_1^c)\mathfrak{f}\rangle\rangle + \langle\langle \mathfrak{g}, (\lambda - \mathbb{S} - \mathbb{L}_1^c)\mathfrak{g}\rangle\rangle \Big\}.
\end{aligned}$$

Let a_n denote the supremum restricted to finite supported functions in $\mathfrak{f}$ in $L^2(\mathcal{G}_n)$, and a^n denote the infimum restricted to finite supported function $\mathfrak{g}$ in $L^2(\mathcal{G}_n)$ so that $a_n \uparrow \langle\langle \sigma, (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\sigma\rangle\rangle$ and $a^n \downarrow \langle\langle \sigma, (\lambda - \mathcal{L}^{ex} - \mathcal{L}_1^c)^{-1}\sigma\rangle\rangle$. By straightforward computation one checks that $a_n \leqslant \langle\langle \sigma, (\lambda - \mathcal{L}_{n+1})^{-1}\sigma\rangle\rangle \leqslant a^n$, giving the desired result. ∎

In what follows we will present a general approach to the Eqs. (4.2) which, from (4.4) gives a nontrivial lower bound on the diffusion coefficient. Because it gives a sequence of upper and lower bounds, the method has the potential to give the full conjectured scaling of the diffusion coefficient.

5. REMOVAL OF HARD CORE

From Lemma 4.1 of the previous section we have a lower bound at degree three. However, computations are complicated due to the hard core exclusion. We follow Ref. 11 to remove the hard core restriction in the formulas and then perform explicit computations in Fourier space. By removal of the hard core, we mean replacing functions defined on $\mathcal{E}_n$ by symmetric functions defined on $E_n = (\mathbb{Z}^d \times \mathcal{V})^n$ and replacing operators acting on $\mathcal{E}_n$ by operators acting on E_n.

We first identify a function $f: \mathcal{E}_n \to \mathbb{R}$ with a symmetric function $f: E_n \to \mathbb{R}$. Denote by $\boldsymbol{\omega}_n = (\omega_1, \dots, \omega_n)$, $\omega_i = (x_i, v_i)$, the points of E_n. For $n \geqslant 1$, let

$$E_{n,1} = \{\boldsymbol{\omega}_n : \omega_i \neq \omega_j, \text{ for } i \neq j\} \tag{5.1}$$

and define

$$f(\boldsymbol{\omega}_n) = \begin{cases} f(\{\omega_1, \dots, \omega_n\}) & \text{if } \boldsymbol{\omega}_n \in E_{n,1}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

With the notation just introduced,

$$E_{\mu_0}\left[\left(\sum_{A \in \mathcal{E}_n} \mathfrak{f}_A \Psi_A\right)^2\right] = \frac{1}{n!\,4^n} \sum_{\boldsymbol{\omega}_n \in E_n} \mathfrak{f}(\boldsymbol{\omega}_n)^2\,.$$

For a function $f: E_n \to \mathbb{R}$, we shall use the same symbol $\langle f \rangle$ to denote the expectation

$$\frac{1}{n!\,4^n} \sum_{\boldsymbol{\omega}_n \in E_n} f(\boldsymbol{\omega}_n)$$

and write the inner product of two functions as $\langle f, g \rangle = \langle f g \rangle$. If f and g vanish on the complement of $E_{n,1}$, this coincides with the inner product introduced before. We also define, as before, $\langle\langle f, g \rangle\rangle = \sum_{x \in \mathbb{Z}^d} \langle \tau_x f, g \rangle$.

Let $E = \cup_{n \geqslant 1} E_n$, $G_n = \cup_{1 \leqslant j \leqslant n} E_j$. We use the same symbol π_n, $P_n = \sum_{1 \leqslant j \leqslant n} \pi_j$ for the projection onto E_n, G_n. As before, there is a simple formula for the inner product $\langle\langle \cdot, \cdot \rangle\rangle$. Consider two finitely supported functions f, $g: E_n \to \mathbb{R}$. By definition,

$$n!\,4^n \langle\langle f, g \rangle\rangle = \sum_{\substack{\boldsymbol{\omega}_n \in E_n \\ z \in \mathbb{Z}^d}} f(\boldsymbol{\omega}_n + z) g(\boldsymbol{\omega}_n) = \sum_{\boldsymbol{\omega}_n \in E_n} \tilde{f}(\boldsymbol{\omega}_n) g(\boldsymbol{\omega}_n)\,,$$

where $\boldsymbol{\omega}_n + z = ((x_1 + z, v_1), \dots, (x_n + z, v_n))$.

Denote by $\sim$ the equivalence relation on E_n defined by $\boldsymbol{\omega}_n \sim \boldsymbol{\omega}'_n$ if for all $1 \leqslant i \leqslant n$, $v_i = v'_i$, $x_i - x'_i = z$ for some z in $\mathbb{Z}^d$. Let $\tilde{E}_n = E_n\big|_{\sim}$. Since summing over all sites in E_n is the same as summing over all equivalence classes and then over all elements of a single class, the previous sum is equal to

$$\sum_{\boldsymbol{\omega}_n \in \tilde{E}_n} \sum_{z \in \mathbb{Z}^d} \tilde{f}(\boldsymbol{\omega}_n + z) g(\boldsymbol{\omega}_n + z) = \sum_{\boldsymbol{\omega}_n \in \tilde{E}_n} \tilde{f}(\boldsymbol{\omega}_n) \tilde{g}(\boldsymbol{\omega}_n)$$

because $\tilde{f}(\boldsymbol{\omega}_n + z) = \tilde{f}(\boldsymbol{\omega}_n)$. It remains to choose an element of each class. This can be done by fixing the last coordinate x_n to be zero. In conclusion,

$$n!\,4^n \langle\langle f, g\rangle\rangle = \sum_{v \in \mathcal{V}} \sum_{\boldsymbol{\omega}_{n-1} \in E_{n-1}} f^*(\boldsymbol{\omega}_{n-1}, v) g^*(\boldsymbol{\omega}_{n-1}, v)\,, \tag{5.2}$$

where

$$f^*(\boldsymbol{\omega}_{n-1}, v) = \sum_{z \in \mathbb{Z}^d} f((x_1 + z, v_1), \ldots, (x_{n-1} + z, v_{n-1}), (z, v))\,. \tag{5.3}$$

Here again we see that the translations in the inner product effectively reduce the degree of a function by one.

We derive now explicit formulas for the operators $\mathbb{S}$, $\mathbb{A}_+$ acting on symmetric functions of E_n. An elementary computation shows that

$$(\mathbb{S} f)(\boldsymbol{\omega}_n) = \gamma \sum_{k=1}^{d} \sum_{i=1}^{n} \sum_{\iota = \pm} \mathbf{1}\{\nabla^{\iota}_{k,i} \boldsymbol{\omega}_n \in E_{n,1}\} \nabla^{\iota}_{k,i} f(\boldsymbol{\omega}_n)$$

if $\boldsymbol{\omega}_n$ belongs to $E_{n,1}$ and $(\mathbb{S} f)(\boldsymbol{\omega}_n) = 0$ if $\boldsymbol{\omega}_n$ does not. Here, for $\iota = \pm$,

$$\begin{aligned}
\nabla^{\iota}_{k,i} \boldsymbol{\omega}_n &= (\omega_1, \ldots, \omega_{i-1}, (x_i + \iota e_k, v_i), \omega_{i+1}, \ldots, \omega_n)\,, \\
\nabla^{\iota}_{k,i} f(\boldsymbol{\omega}_n) &= f(\nabla^{\iota}_{k,i} \boldsymbol{\omega}_n) - f(\boldsymbol{\omega}_n)\,.
\end{aligned}$$

Note that $\mathbb{S}$ is the discrete Laplacian with Neumann boundary condition on $E_{n,1}$. In the same way,

$$(\mathbb{J}_+ f)(\boldsymbol{\omega}_n) = \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k=1}^{d} (e_k \cdot v_i) \mathbf{1}\{x_j + e_k = x_i,\, v_j = v_i\} \nabla^{i,j}_{+} f(\boldsymbol{\omega}_n) \tag{5.4}$$

if $\boldsymbol{\omega}_n$ belongs to $E_{n,1}$ and $(\mathbb{J}_+ f)(\boldsymbol{\omega}_n) = 0$ otherwise. Here,

$$\nabla_+^{i,j} f(\boldsymbol{\omega}_n) = f(\boldsymbol{\omega}_n^i) - f(\boldsymbol{\omega}_n^j)$$

and the index j in $\boldsymbol{\omega}_n^j$ indicates the absence of ω_j in the vector $\boldsymbol{\omega}_n$: $\boldsymbol{\omega}_n^j = (\omega_1, \dots, \omega_{j-1}, \omega_{j+1}, \dots, \omega_n)$. Finally, notice that

$$\begin{aligned}(\mathbb{L}_1^c f)(\boldsymbol{\omega}_n) = (1/4) \sum_{q \in \mathcal{Q}} \sum_{j=1}^{n} i_q(v_j) \mathbf{1}\{x_k \neq x_j \ \text{ for } k \neq j\} \\ [f(\sigma_{j,v'}\boldsymbol{\omega}_n) + f(\sigma_{j,w'}\boldsymbol{\omega}_n) - f(\sigma_{j,v}\boldsymbol{\omega}_n) - f(\sigma_{j,w}\boldsymbol{\omega}_n)],\end{aligned}$$

if $\boldsymbol{\omega}_n$ belongs to $E_{n,1}$ and $(\mathbb{L}_1^c f)(\boldsymbol{\omega}_n) = 0$ otherwise. Here, $\sigma_{j,u}\boldsymbol{\omega}_n = (\omega_1, \dots, \omega_{j-1}, \ (x_j, u), \omega_{j+1}, \dots, \omega_n)$.

We now extend the operators $\mathbb{S}$, $\mathbb{J}_+$ to symmetric functions not necessarily vanishing on $E_{n,1}$ by formulas analogous to the ones above, except that we drop some indicator functions. Let S, J_+ and L_1^c be the operators defined by:

$$\begin{aligned}(SF)(\boldsymbol{\omega}_n) &= \gamma(\Delta F)(\boldsymbol{\omega}_n) = \gamma \sum_{i=1}^{n} \sum_{\iota = \pm} \sum_{k=1}^{d} (\nabla_{k,i}^{\iota} F)(\boldsymbol{\omega}_n), \\ (J_+ F)(\boldsymbol{\omega}_n) &= \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k=1}^{d} (e_k \cdot v_i) \mathbf{1}\{x_j + e_k = x_i, \, v_i = v_j\} \nabla_+^{i,j} F(\boldsymbol{\omega}_n) \quad \text{and} \\ (L_1^c f)(\boldsymbol{\omega}_n) &= (1/4) \sum_{q \in \mathcal{Q}} \sum_{j=1}^{n} i_q(v_j) \mathbf{1}\{x_k \neq x_j \ \text{ for } k \neq j\} \\ &\qquad [f(\sigma_{j,v'}\boldsymbol{\omega}_n) + f(\sigma_{j,w'}\boldsymbol{\omega}_n) - f(\sigma_{j,v}\boldsymbol{\omega}_n) - f(\sigma_{j,w}\boldsymbol{\omega}_n)].\end{aligned}$$

Notice that $\langle L_1^c F \rangle = \langle J_+ F \rangle = 0$ if $\langle |F| \rangle < \infty$ and hence the counting measure is invariant. Let

$$L = S + L_1^c + J_+ .$$

and denote by $L_n = P_n L P_n$ the restriction of L to G_n. Following Section 4 in Ref. 11, we prove the next result which permits to avoid the hard core interaction of the exclusion.

Proposition 5.1. In dimension $d=2$, there exists a finite constant C_0 such that

$$\frac{1}{C_0 n^6}\langle\langle \sigma, (\lambda - L_n)^{-1}\sigma\rangle\rangle \leqslant \langle\langle \sigma, (\lambda - \mathcal{L}_n)^{-1}\sigma\rangle\rangle \leqslant C_0 n^4 \langle\langle \sigma, (\lambda - L_n)^{-1}\sigma\rangle\rangle\,.$$

for all $\lambda > 0$.

The proof of this proposition is similar to the one of Lemma 3.1 in Ref. 11 and therefore omitted. The main difference is to prove Lemma 4.3 in Ref. 11 with $S + L_1^c$, $\mathbb{S} + \mathbb{L}_1^c$ in place of S, $\mathbb{S}$ and this is elementary.

The special case $n=3$ combined with Lemmas 3.1 and 4.1 tells us that

$$\langle\langle \sigma, (\lambda - \mathcal{L})^{-1}\sigma\rangle\rangle \geqslant C_0 \langle\langle \sigma, (\lambda - L_3)^{-1}\sigma\rangle\rangle\,.$$

6. FOURIER COMPUTATIONS

To bound below $\langle\langle \sigma, (\lambda - L_3)^{-1}\sigma\rangle\rangle$, define the Fourier transform of a function $\mathfrak{f}\colon (\mathbb{Z}^d \times \mathcal{V})^n \to \mathbb{R}$ by

$$\widehat{\mathfrak{f}}(\boldsymbol{p}_n, \boldsymbol{v}_n) = \sum_{\boldsymbol{x}_n \in \mathbb{Z}^{nd}} e^{-i\boldsymbol{x}_n\cdot\boldsymbol{p}_n}\mathfrak{f}(\boldsymbol{x}_n, \boldsymbol{v}_n)$$

for $\boldsymbol{p}_n \in (\mathbb{R}^d/2\pi\mathbb{Z}^d)^n$. Here we represented $\boldsymbol{w}_n = (w_1, \ldots, w_n)$, $w_i = (x_i, v_i)$, as $(\boldsymbol{x}_n, \boldsymbol{v}_n)$ with $\boldsymbol{x}_n = (x_1, \ldots, x_n) \in (\mathbb{Z}^d)^n$, $\boldsymbol{v}_n = (v_1, \ldots, v_n) \in \mathcal{V}^n$.

An elementary computation together with (5.2) shows that for any local functions f, g of degree n,

$$\langle\langle f, g\rangle\rangle = \frac{1}{(2\pi)^{(n-1)d} n!\, 4^n} \sum_{\boldsymbol{v}_n \in \mathcal{V}^n} \int_{[-\pi,\pi]^{(n-1)d}} \widehat{\mathfrak{f}^*}(\boldsymbol{p}_{n-1}, \boldsymbol{v}_n)\widehat{\mathfrak{g}^*}(\boldsymbol{p}_{n-1}, \boldsymbol{v}_n)\, d\boldsymbol{p}_{n-1}\,.$$

In this formula, $\mathfrak{f} = \mathbb{T}f$, $\mathfrak{f}^*$ is defined by (5.3) and $\widehat{\mathfrak{f}^*}$ is the Fourier transform of $(\mathfrak{f}^*)(\boldsymbol{x}_{n-1}, \boldsymbol{v}_n)$. Expressing $\widehat{\mathfrak{f}^*}$ in terms of $\widehat{\mathfrak{f}}$ we further obtain that

$$\langle\langle f, g\rangle\rangle = \frac{1}{(2\pi)^{(n-1)d} n!\, 4^n} \sum_{\boldsymbol{v}_n \in \mathcal{V}^n} \int_{\substack{[-\pi,\pi]^{nd} \\ \sum_{1\leqslant j\leqslant n} p_j = 0}} \widehat{\mathfrak{f}}(\boldsymbol{p}_n, \boldsymbol{v}_n)\widehat{\mathfrak{g}}(\boldsymbol{p}_n, \boldsymbol{v}_n)\, d\boldsymbol{p}_n\,.$$

Fix a symmetric function $\mathfrak{f}\colon (\mathbb{Z}^d \times \mathcal{V})^n \to \mathbb{R}$. The Fourier transform of the discrete Laplacian acting on $\mathfrak{f}$ is given by

$$-\widehat{\Delta \mathfrak{f}}(\boldsymbol{p}_n, \boldsymbol{v}_n) = \widehat{\mathfrak{f}}(\boldsymbol{p}_n, \boldsymbol{v}_n) W(\boldsymbol{p}_n)\,,$$

where

$$W(\boldsymbol{p}_n) = \sum_{j=1}^{n} W(p_j) = \sum_{j=1}^{n} \sum_{k=1}^{d} \{1 - \cos(p_j \cdot e_k)\}$$

if $\boldsymbol{p}_n = (p_1, \ldots, p_n)$. Notice that we are using the same notation $W(\cdot)$ for slightly different objects. Moreover, for $n = 2$, a straightforward computation shows that

$$\begin{aligned}\widehat{J_+ \mathfrak{f}}\,(p_1, p_2, p_3, \boldsymbol{v}_3) = -i \sum_{j=1}^{d} \sum_{\sigma} \mathbf{1}\{v_{\sigma_1} = v_{\sigma_2}\}\,(e_j \cdot v_{\sigma_1}) \\ \times \{\sin(e_j \cdot p_{\sigma_1}) + \sin(e_j \cdot p_{\sigma_2})\}\,\widehat{\mathfrak{f}}(p_{\sigma_1} + p_{\sigma_2}, p_{\sigma_3}; v_{\sigma_1}, v_{\sigma_3}),\end{aligned}$$

where σ runs over all permutations of degree three.

We are now in a position to state the first estimate based on Fourier arguments.

Lemma 6.1. Fix a symmetric function $\mathfrak{f}\colon (\mathbb{Z}^d \times \mathcal{V})^2 \to \mathbb{R}$. There exists a finite constant C_0 such that in dimension 2,

$$\begin{aligned}&\langle\langle J_+ \mathfrak{f}, (\lambda - \Delta_3)^{-1} J_+ \mathfrak{f} \rangle\rangle \\ &\quad \leqslant C_0 \sum_{v_1, v_2 \in \mathcal{V}} \int_{[-\pi,\pi]^2} dp\, W(p)\, |\log(\lambda + W(p))|\, |\widehat{\mathfrak{f}}(p, -p; v_1, v_2)|^2\,.\end{aligned}$$

The proof of this result is similar to the one of Lemma 3.2 in Ref. 11 and therefore omitted.

For $0 \leqslant j \leqslant 2$, let $\mathbb{I}_j = \mathbb{T} I_j$, $0 \leqslant j \leqslant 2$ be the symmetric functions in E_n associated to the conserved quantities. An elementary computation shows that

$$\mathbb{I}_0(v) = 1\,, \quad \mathbb{I}_1(v) = e_1 \cdot v\,, \quad \mathbb{I}_2(v) = e_2 \cdot v\,.$$

Let Q be the non positive symmetric matrix corresponding to the operator L_1^c acting on functions depending only on one site. Notice that

$\mathbb{I}_a$, $a=0$, 1, 2, are eigenvectors of Q. Denote by $\{\mathbb{I}_a,\ 3\leqslant a\leqslant|\mathcal{V}|-1\}$ the other eigenvectors of Q and by $\{q_a,\ 0\leqslant a\leqslant|\mathcal{V}|-1\}$ the corresponding eigenvalues. Since $\mathbb{I}_0$, $\mathbb{I}_1$, $\mathbb{I}_2$ are associated to conserved quantities, $q_0=q_1=q_2=0$. Let $\mathcal{O}$ be the orthogonal matrix which diagonalizes Q:

$$\mathcal{D}=\mathcal{O}^*Q\mathcal{O}\,.$$

For $n\geqslant 1$, let

$$Q_n=\sum_{i=1}^{n} I\otimes\cdots\otimes Q\otimes\cdots\otimes I\,.$$

Since L_1^c has an indicator function, we have that $0\leqslant -L^c\leqslant -Q_n$. Moreover, Q_n can be diagonalized by $\mathcal{O}_n=\mathcal{O}^{\otimes n}$: $\mathcal{D}_n=\mathcal{O}_n^*Q_n\mathcal{O}_n$, where $\mathcal{D}_n=\sum I\otimes\cdots\otimes\mathcal{D}\otimes\cdots\otimes I$. Notice that the Laplacian commutes with these matrices.

As in section 4, we can represent $(\lambda-L_3)^{-1}\sigma$ in terms of the operators S, J_+ and L_1^c to obtain that

$$\langle\langle\sigma,(\lambda-L_3)^{-1}\sigma\rangle\rangle=\langle\langle\sigma,\left\{\lambda-\Delta-L_1^c+J_+^*(\lambda-\Delta-L_1^c)^{-1}J_+\right\}^{-1}\sigma\rangle\rangle\,.$$

Since $0\leqslant -L_c^1\leqslant -Q_n$, the previous scalar product is bounded below by

$$\langle\langle\sigma,\left\{\lambda-\Delta_2-Q_2+J_+^*(\lambda-\Delta_3)^{-1}J_+\right\}^{-1}\sigma\rangle\rangle\,.$$

Recalling that $Q_2=\mathcal{O}_2\mathcal{D}_2\mathcal{O}_2^*$ and that Δ commutes with the operators $\mathcal{O}_2$, we rewrite the previous expression as

$$\langle\langle\mathcal{O}_2^*\sigma,\left\{\lambda-\Delta_2-\mathcal{D}_2+\mathcal{O}_2^*J_+^*(\lambda-\Delta_3)^{-1}J_+\mathcal{O}_2\right\}^{-1}\mathcal{O}_2^*\sigma\rangle\rangle\,.$$

To keep notation simple, let $\Omega=\{\lambda-\Delta_2-\mathcal{D}_2+\mathcal{O}_2^*J_+^*(\lambda-\Delta_3)^{-1}J_+\mathcal{O}_2\}^{-1}$ and denote by $\bar\pi$ the projection onto the eigenspace corresponding to the zero eigenvalue of $\mathcal{D}_2$. By Schwarz inequality, for any function H,

$$\langle\langle H,\Omega H\rangle\rangle\geqslant(1/2)\langle\langle\bar\pi H,\Omega\bar\pi H\rangle\rangle-\langle\langle(1-\bar\pi)H,\Omega(1-\bar\pi)H\rangle\rangle\,.$$

We claim that $\langle\langle(1-\bar\pi)\mathcal{O}_2^*\sigma,\Omega(1-\bar\pi)\mathcal{O}_2^*\sigma\rangle\rangle$ is bounded by a finite constant. Indeed, by definition of Ω,

$$\langle\langle(1-\bar\pi)\mathcal{O}_2^*\sigma,\Omega(1-\bar\pi)\mathcal{O}_2^*\sigma\rangle\rangle\leqslant\langle\langle(1-\bar\pi)O_2^*\sigma,(-\mathcal{D}_2)^{-1}(1-\bar\pi)\mathcal{O}_2^*\sigma\rangle\rangle\,.$$

Since $(1-\bar{\pi})$ is the projection on the positive eigenvalues of $\mathcal{D}_2$, the previous expression is less than or equal to a finite constant which depends on a lower bound for the positive eigenvalues of Q and an upper bound for σ.

The following lemma is needed to estimate $\langle\langle \bar{\pi}\mathcal{O}_2^*\sigma, \Omega\bar{\pi}\mathcal{O}_2^*\sigma\rangle\rangle$. Recall that $\sigma = \sum_{0\leqslant a\leqslant d}\sum_{1\leqslant j\leqslant d} r_a\theta_j\sigma_j^a$.

Lemma 6.2. $\bar{\pi}\mathcal{O}_2^*\sigma = 0$ if and only if $\theta = 0$ or $r = 0$.

Proof. Assume that $\bar{\pi}\mathcal{O}_2^*\sigma = 0$ and assume without loss of generality that $\theta_1 \neq 0$. Fix the component $\{0, e_1\}$. Since $\bar{\pi}$ is the projection of the eigenspace of $\mathcal{D}_2$ associated to the zero eigenvalues, $\bar{\pi}\mathcal{O}_2^*\sigma = 0$ if and only if the scalar product of σ with $(\mathbb{I}_a, \mathbb{I}_b)$ vanishes for $0\leqslant a, b\leqslant 2$.

An elementary computation based on the explicit expression (4.1) for σ shows that

$$\sigma\cdot(\mathbb{I}_a, \mathbb{I}_b) = \theta_1 \sum_{v\in\mathcal{V}} \mathbb{I}_a(v)\,\mathbb{I}_b(v)\big\{r_0(e_1\cdot v) + r_1(e_1\cdot v)^2 + r_2(e_1\cdot v)(e_2\cdot v)\big\}.$$

Setting $a = 0$ and $b = 1$ we get that

$$\theta_1 \sum_{v\in\mathcal{V}} \big\{r_0(e_1\cdot v)^2 + r_1(e_1\cdot v)^3 + r_2(e_1\cdot v)^2(e_2\cdot v)\big\}.$$

The last two terms vanish due to the symmetry of $\mathcal{V}$. Hence $r_0 = 0$. Repeating the same argument with $a = b = 1$ (resp. $a = 1, b = 2$), we obtain that $r_1 = 0$ (resp. $r_2 = 0$). This concludes the proof of the lemma. ∎

We are now in a position to bound below

$$\langle\langle \bar{\pi}\mathcal{O}_2^*\sigma, \{\lambda - \Delta_2 - \mathcal{D}_2 + \mathcal{O}_2^* J_+^*(\lambda-\Delta_3)^{-1} J_+\mathcal{O}_2\}^{-1}\bar{\pi}\mathcal{O}_2^*\sigma\rangle\rangle.$$

Fix a function G in the image of $\bar{\pi}$. By the variational formula for the H_{-1} norm, the previous scalar product with G in place of $\bar{\pi}\mathcal{O}_2^*\sigma$ is equal to

$$\sup_F \Big\{2\langle\langle G, F\rangle\rangle - \langle\langle F, [\lambda-\Delta_2-\mathcal{D}_2]F\rangle\rangle - \langle\langle \mathcal{O}_2 F, J_+^*(\lambda-\Delta_3)^{-1} J_+\mathcal{O}_2 F\rangle\rangle\Big\},$$

where the supremum is performed over all local functions F. Restricting this supremum to functions in the image of $\bar{\pi}$ we obtain a lower bound. In this case we may remove the operator $\mathcal{D}_2$ because $\bar{\pi}$ is the projection onto the space associated to the zero eigenvalues of $\mathcal{D}_2$.

Lemma 6.1 and a straightforward computation show that

$$\langle\langle \mathcal{O}_2 F, J_+^*(\lambda - \Delta_3)^{-1} J_+ \mathcal{O}_2 F \rangle\rangle$$

is bounded above by the right hand side of the statement of Lemma 6.1. Now, repeating the arguments presented in the proof of Lemma 3.3 in Ref. 11, we obtain that the previous variational formula is bounded below by

$$C_0 \sum_{\boldsymbol{v}_2} \int_{[-\pi,\pi)^2} \frac{|\widehat{G}(p,-p,\boldsymbol{v}_2)|^2}{\lambda + 2W(p) + C_1 W(p)|\log\{\lambda + W(p)\}|}\, dp$$

for some finite constants C_0, C_1. We used here that G is in the image of $\bar{\pi}$ because we needed the function F which maximizes the supremum to be in the image of $\bar{\pi}$. The factor 2 appeared because $W(p,-p) = 2W(p)$.

Replace G by $\bar{\pi}\mathcal{O}_2^*\sigma$. It follows from the previous lemma and elementary computations that $\sum_{\boldsymbol{v}_2} |\widehat{\bar{\pi}\mathcal{O}_2^*\sigma}(p,-p,\boldsymbol{v}_2)|^2$ is bounded below by a positive constant close to the origin. The previous expression is therefore bounded below by

$$C_0 \int_{[-\epsilon,\epsilon)^2} \frac{1}{\lambda + 2W(p) + C_1 W(p)|\log\{\lambda + W(p)\}|}\, dp$$

for some $\epsilon > 0$. Changing to polar coordinates, since there is no angle dependence, we get an integral of the form

$$\int_0^\rho \frac{r dr}{\lambda + r^2(2 - C_1 \log(\lambda + r^2))} \sim \int_0^\rho \frac{-r\, dr}{(\lambda + r^2)\log(\lambda + r^2)}$$

for some $\rho > 0$. By the change of variables $u = -\log(\lambda + r^2)$ we finally get

$$\sim \int_{-\log \rho^2}^{-\log \lambda} \frac{du}{u} \sim \log\log \lambda^{-1}\,.$$

This proves Lemma 2.2.

Recall the recursive relation (4.3). If we denote the limit of $\mathcal{T}_n = \mathcal{T}$, then it satisfies the equation

$$\mathcal{T} = \left[(\lambda - \mathbb{S} - \mathbb{L}_1^c) + \mathbb{J}_- \mathcal{T}^{-1} \mathbb{J}_+\right]^{-1}.$$

We now assume that the dispersion relation of $\mathcal{T}^{-1}$ is given by

$$\left\{\sum_{j=1}^{\infty} W(p_j)\right\} \left|\log\left(\lambda+\left\{\sum_{j=1}^{\infty} W(p_j)\right\}\right)\right|^{\kappa}$$

for some $\kappa > 0$. Write $\sum_{j=1}^{\infty} W(p_j) = u + W(p_1)$ where $u = \sum_{j=2}^{\infty} W(p_j)$. Suppose that we are interested in $u \geqslant \lambda$. Then $\mathbb{J}_{-}\mathcal{T}^{-1}\mathbb{J}_{+}$ is approximately given by

$$\int_{[-\epsilon,\epsilon)^2} \frac{1}{u+W(p)+(u+W(p))|\log\{u+W(p)\}|^{\kappa}}\, dp \sim u|\log u|^{1-\kappa} \tag{6.1}$$

For $u \geqslant \lambda$ we have

$$\mathcal{T}^{-1} = (\lambda - \mathbb{S} - \mathbb{L}_1^c) + \mathbb{J}_{-}\mathcal{T}^{-1}\mathbb{J}_{+} \sim \mathbb{J}_{-}\mathcal{T}^{-1}\mathbb{J}_{+}$$

Thus we have $\kappa = 1-\kappa$ and this gives the value $\kappa = 1/2$. This is precisely the exponent derived in Refs. 2, 5.

Notice that the exponent is different from the ASEP which takes the value $\kappa = 2/3$.[15] This is because that the dispersion law is changed only in one direction for ASEP. In the fluid model considered here, the collision operator spread the dispersion law to all direction. This causes the exponent on the right side of (6.1) to be $1-\kappa$, compared with $1-\kappa/2$ in Ref. 15. This sketch is certainly not rigorous since it is not even clear where the operator $\mathcal{T}$ should be defined. Though one can try to prove upper and lower bound as in Ref. 15, it is not clear that the off-diagonal terms can be controlled. However, the exponent $1/2$ seems to be very convincing.

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